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CALCULATING BLADE RING FLUTTER

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The investigation of elastic blade ring stability to small vibrations in a flow is one of the basic problems of turbomachinery aeroelasticity. In recent decades, primary attention in this field has been paid to the development of effective methods for calculating nonstationary aerodynamic forces on a cascade of vibrating blades. Nevertheless, despite considerable success in this field, the numerical investigation of the boundary of the domain in the space of cascade and flow parameters for which self-excited vibration for the blade ring of a turbomachine is possible, in general, presents major difficulties pertaining to the nonconservative character of aerodynamic forces. One of the most common approximate models is based on the smallness of aerodynamic forces acting on the blade, compared to the elastic force [1]. For identical blades, this assumption formally leads to the characteristic number for an aeroelastic system coinciding with the corresponding aerodynamic coefficient determined for the natural frequency of blade vibrations in vacuum.

Questions related to the applicability of this model and to its use in the case of a small spread in natural frequencies of blade have not been investigated sufficiently, despite their practical importance. These are the questions to which this paper is devoted. Its results are obtained by developing a method for calculating the boundaries of self-excited vibrations of a blade ring in an axial compressor, within the space of its parameters. To estimate the influence of the relationship between aerodynamic and elastic forces on the characteristic number of an aeroelastic system, we propose an aeroelastic model which consists of a system of clamped thin-walled rods in the incompressible flow described by the asymptotic theory of high-solidity airfoil cascades [2]. To analyze the stability of the ring formed by blades with a small dispersion of natural frequencies, we use the stability quality parameter for the solution of a linear system that was studied in [3, 4]. The calculated results are compared with the experimental results obtained on a compression test bed of the Central Institute of Aviation Engineering.

Let a ring of Z vibrating blades with numbers $k = 0, 1, \ldots, Z-1$ with given vibration forms φ, generalized masses M_k, and rigidity Q_k be put in an annular channel with peripherally uniform inviscid flow (Fig. 1). The blades are rigidly fastened in an inner body (in a disk), and links between them are determined by generalized aerodynamic coefficients of influence [1] or coefficients of generalized aerodynamic forces corresponding to a given form of blade vibrations. Structural and material damping is neglected. The task is to determine the flow parameters at which the blade vibrations do not damp.

The equation of small vibrations for the described aeroelastic system in the absence of external forces has the form

$$M_k\ddot{x}_k(t) + Q_k x_k(t) - \sum_{l=0}^{Z-1} A_{l-k}x_l(t) = 0, \qquad k = 0, 1, \ldots, Z-1, \tag{1}$$

where t is time; $\mathbf{X} = (x_0, x_1, \ldots, x_{Z-1})$ is the vector of generalized displacements of the blades vibrating according to the form φ; $A_{l-k} = A_r (A_r = A_{Z-r}, r = 0, 1, \ldots, Z-1)$ are the aerodynamic influence coefficients (AIC) of a blade with number r on the initial blade ($r = 0$).

The values of A_r depend on the geometrical parameters of the ring, the inflow parameters, and the forms of blade vibrations, as well as on the reduced frequencies of blade vibrations (Strouhal number) $\mathrm{Sh} = \Omega b/V_1$. Here Ω is the ring vibration frequency, 1/sec, b is the characteristic linear size of the blade (length of the

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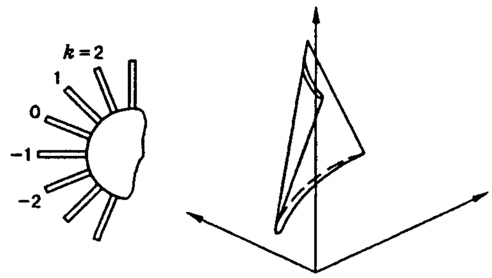

Fig. 1

blade chord in a peripheral cross section); V_1 is the velocity of the relative flow at the entrance to the ring in the peripheral cross section. If a nontrivial solution of homogeneous system (1) is represented as

$$\mathbf{X}(t) = \overline{\mathbf{X}} e^{i\lambda t}, \qquad i = \sqrt{-1}, \tag{2}$$

where $\overline{\mathbf{X}} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{Z-1})$ is an eigenvector, then, at the boundary of self-excited vibrations that corresponds to the condition $\lambda = \operatorname{Re}\lambda$, the Strouhal number is determined by the frequency $\Omega = \lambda$. In this case, the aerodynamic influence coefficients are complex. They are the most complicated element of system (1), and success in calculating the boundary of the ring self-excited vibrations depends on the possibility of solving the corresponding gasdynamic problem.

The proposed aeroelastic model of the blade ring can be basically simplified if the following assumptions are adopted [1]:

(1) elastic forces in the blades considerably exceed nonstationary aerodynamic forces affecting the blades;

(2) natural frequencies of the blade vibrations in vacuum $\omega_k (k = 0, 1, \ldots, Z-1)$ differ little from their mean square values

$$\Omega = \left\{ \frac{1}{Z} \sum_{k=0}^{Z-1} \omega_k^2 \right\}^{1/2}.$$

Hence, Eq. (1), after substitution of solution (2) in it, can be written in the form

$$(\Delta\omega_k - \nu)\bar{x}_k - R \sum_{l=0}^{Z-1} A_{l-k}\bar{x}_l + O(R^2, R|M - M_k|) = 0. \tag{3}$$

Here

$$\Delta\omega_k = \frac{\omega_k^2 - \Omega^2}{\Omega^2}; \quad \nu = \frac{\lambda^2 - \Omega^2}{\Omega^2}; \quad R = \frac{1}{2}\frac{\rho_g V_1^2 b}{\Omega^2 M}; \quad \omega_k = \frac{Q_k}{M_k}; \quad M = \frac{1}{Z}\sum_{k=0}^{Z-1} M_k;$$

ρ_g is the characteristic density of the flow. The influence coefficient A_{l-k} related to the velocity head retains its initial notation.

Equation (3) shows that, at conditions $R \ll 1$ (assumption 1) and $\max\limits_{0 \leqslant k \leqslant Z-1} \Delta\omega_k \ll 1$ (assumption 2), the values of ν for which there is a nonzero solution have the order of smallness of values of R and $\max\limits_{0 \leqslant k \leqslant Z-1} \Delta\omega_k$. From this it follows that, with accuracy up to the linear terms in Eq. (3), it may be assumed that $A_{l-k}(\mathrm{Sh}) = A_{l-k}(\mathrm{Sh}_0)$, $\mathrm{Sh}_0 = \Omega b / V_1$.

Thus, in the case under consideration, determination of the ring self-excited vibration boundary is reduced to consecutively solving problems of the elastic ring dynamics in vacuum, the aerodynamics of ring blades vibrating at a given frequency and form, and the stability of solution for a linear system of first-order equations.

For arbitrary values of the parameter R, aerodynamic forces in Eq. (1) are inseparable from forces of inertia and elasticity. In particular, for identical blades ($M_k = M$, $Q_k = Q$, $k = 0, 1, \ldots, Z-1$) oscillating

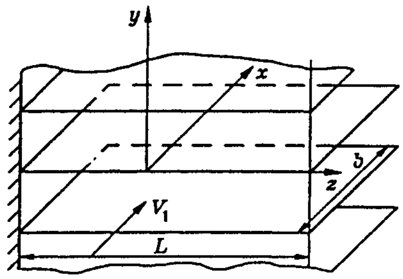

Fig. 2

under law (2) with a given form φ, Eq. (1), after multiplying it by the value b/V_1^2, takes the form

$$(\mathrm{Sh}_0^2 - \mathrm{Sh}^2)\bar{x}_k - \frac{1}{2}\bar{R}\sum_{l=0}^{Z-1} A_{l-k}(\mathrm{Sh})\bar{x}_l = 0, \qquad k = 0, 1, \ldots, Z-1, \tag{4}$$

where $\bar{R} = \rho_g b^3/M$ is the reduced mass criterion, and the reduced frequency $\mathrm{Sh}_0 = \sqrt{Q/M}(b/V_1)$ is determined by the natural frequency of the blade vibrations in vacuum.

The boundary of vibrations of the aeroelastic system corresponds to real values $\mathrm{Sh} = \mathrm{Sh}^*$ that cause the determinant of the matrix of system (4) to vanish, and the corresponding complex eigenvectors $\overline{\mathbf{X}} = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{Z-1})$ give a relative distribution of the generalized displacements of the blades.

Solving the eigenvalue problem for system (4) in the general case is very difficult due to the transcendental dependence of AIC on the generalized frequency Sh. For the model of flow past a cascade, this dependence is simpler, and an analytical solution of the problem is possible.

Let a straight cascade consist of thin, inflexibly fixed plates with thickness Δ, width b, and span L (Fig. 2). We shall assume that Δ/d, b/L, $h/b \ll 1$ (h is the cascade pitch). We will consider only bending vibrations of the cascade in an incompressible flow, whose velocity vector at infinity before the cascade forms an angle α_0 with a normal to the front of the cascade. The equation for small vibrations of the aeroelastic system can be written as

$$\rho_{\mathrm{m}} S \frac{\partial^2 y_k(z,t)}{\partial t^2} + E J_x \frac{\partial^4 y_k(z,t)}{\partial z^4} - \frac{1}{2}\rho_{\mathrm{g}} V_1^2 C_k y_k(z,t) = 0. \tag{5}$$

Here $S = b\Delta$ is the cross-sectional area of the plate, $y = y_k(z,t)$ is the shift along the normal to the middle plane of the plate, E is the module of elasticity; J_x is the moment of inertia, C_k is the dimensionless coefficient of the linear nonstationary aerodynamic load on a plate in the cascade, and ρ_{m} is the density of the plate material.

Since all plates in the cascade are identical, and any one of them can be chosen as the initial one, we shall seek harmonic solutions of Eq. (5) in the form

$$y_k(z,t) = \bar{y}(z)\,\mathrm{e}^{i(\omega t + k\mu)}, \tag{6}$$

where the number μ characterizes the phase shift in the displacements of neighboring plates in the cascade. After substitution of (6) into (5), we obtain the known equation for the vibration form of the thin-walled rods under consideration:

$$\frac{d^4\bar{y}}{dz^4} - \alpha^4\bar{y} = 0. \tag{7}$$

Here

$$\alpha^4 = \frac{\rho_{\mathrm{m}} S\omega^2}{EJ_x}\left[1 + \frac{1}{2}\frac{\rho_{\mathrm{g}} b}{\rho_{\mathrm{m}}\Delta}\frac{V_1^2}{\omega^2 b^2} C(\mathrm{Sh}, \mu)\right] \tag{8}$$

is one of the countable set of values that admit nonzero solutions of Eq. (7) subject to the following boundary conditions: $\bar{y} = \bar{y}' = 0$, $z = 0$ (fixed end), $\bar{y}'' = \bar{y}''' = 0$, $z = 1$ (free end).

The value $\alpha_1 = 1.875/L$ corresponds to the first bending form of the plate vibrations. For values of Sh that provide a nonzero solution for system (5) at $\alpha = \alpha_1$, Eq. (8) leads to the equation

$$\mathrm{Sh}_0^2 - \mathrm{Sh}^2 - \bar{R}C(\mathrm{Sh}, \mu) = 0, \qquad \mathrm{Sh}_0 = \omega_0 b/V_1, \tag{9}$$

where $\omega_0 = \{\alpha_1^4 EJ_x/(\rho_m S)\}^{1/2}$ is the first natural frequency of the vibrations for a plate in vacuum; $\bar{R} = \rho_g b^2/(\rho_m S)$ is the reduced mass criteria. The presence of real values of Sh that satisfy Eq. (9) for the given parameters of the cascade and the flow is equivalent to the existence of undamped vibrations of the cascade of the plates in a flow. The last assertion implies the existence of real values of Sh and μ that satisfy the two equations

$$\mathrm{Im}C(\mathrm{Sh}, \mu) = 0, \qquad \mathrm{Sh}^2 + \mathrm{Re}\,[\bar{R}C(\mathrm{Sh}, \mu)] = \mathrm{Sh}_0^2. \tag{10}$$

Thus, for the reduced frequency Sh^* and the form parameter μ^* of vibrations in an aeroelastic system, any real solution (Sh^*, μ^*) of system (10) gives critical values that correspond to the first bending form of the plate vibrations in the flow.

In explicit form, the function $C(\mathrm{Sh}, \mu)$ can be obtained as an asymptotic solution of the problem of flow past a high-solidity ($\tau = b/h \gg 1$) cascade of small arcs vibrating in an incompressible flow for $|\mu| \ll 1$, $\mu \neq 0$ [2]:

$$C(\mathrm{Sh}, \mu) = \frac{1}{\mu\tau e^{i\beta} + A\,\mathrm{Sh}}\,[A_0 + A_1\mathrm{Sh} + A_2\mathrm{Sh}^2 + A_3\mathrm{Sh}^3].$$

Here β is the angle between the profile chord and the normal to the cascade front, the complex values A, A_0, A_1, A_2, and A_3 depend only on the parameter $\mu\tau$, the inflow angle α_0, the form of the arc, and the form of its vibrations.

Let $\mathrm{Sh}^* = \mathrm{Sh}^*(\mu\tau)$ be a real, positive solution of the first of Eqs. (10). Then

$$d = \mathrm{Re}\left[\frac{1}{(\mathrm{Sh}^*)^2}\,C(\mathrm{Sh}^*, \mu\tau, \alpha_0)\right], \tag{11}$$

where $\bar{R}d = (\omega_0/\omega)^2 - 1$, and ω (the frequency of self-excited vibrations) determines the boundary of the self-excited vibrations of the aeroelastic system under consideration, in the space of the cascade and flow parameters.

For a cascade of inflexibly fixed plates performing only bending vibrations, the function $d = d^*(\alpha_0 - \beta)$ at $\mu\tau = 0.1$ ($\beta = 0$) and 1.0 ($\beta = 45°$) is presented in Fig. 3 (curves 1 and 2), where on the x axis the angle of attack $\alpha = \alpha_0 - \beta$ is plotted. The region of self-excited vibrations corresponds to $d > d^*$.

Note that without aerodynamic load ($\alpha = 0$) the self-excited vibrations considered are impossible. In the region of real values of the reduced mass criteria ($\bar{R} \sim 10^{-2}$–10^{-3}), self-excited vibrations are possible, beginning with some value of the angle of attack ($\alpha > 0$). In addition, for the given values of $\bar{R}$, a decrease in the frequency of self-excited vibrations compared with the natural vibration frequency of the system of plates in vacuum is negligibly small only for sufficiently small values of the aerodynamic characteristic d determined by Eq. (11).

Assuming that, for the blade ring being considered, the value of d is sufficiently small, we can approximate system (3) as

$$(\Delta\omega_k - \nu)\bar{x}_k - R\sum_{l=0}^{Z-1} A_{l-k}\bar{x}_l = 0 \qquad (k = 0,\, 1,\, \ldots,\, Z-1), \tag{12}$$

where $A_{l-k} = A_{l-k}(\mathrm{Sh}_0)$.

Let us multiply (12) by $(1/Z)\,e^{-2\pi ikr/Z}$, $k, r = 0,\, 1,\, \ldots,\, Z-1$, and sum up the obtained equation for all the values of k. After performing elementary transformations, we write system (12) in equivalent form:

$$(RC_r + \nu)\bar{\xi}_r - \sum_{l=0}^{Z-1} \Delta\mu_{l-r}\bar{\xi}_l = 0. \tag{12'}$$

Here ξ_r, C_r, and $\Delta\mu_r$ are related to $\bar{x}_k$, A_k, and $\Delta\omega_k$ by the inversion formulas

$$\bar{\xi}_r = \frac{1}{Z}\sum_{k=0}^{Z-1} \bar{x}_k e^{-2\pi ikr/Z}, \quad C_r = \sum_{k=0}^{Z-1} A_k e^{2\pi ikr/Z}, \quad \Delta\mu_r = \frac{1}{Z}\sum_{k=0}^{Z-1} \Delta\omega_k e^{-2\pi ikr/Z},$$

$$\bar{x}_k = \sum_{r=0}^{Z-1} \bar{\xi}_r e^{2\pi irk/Z}, \quad A_k = \frac{1}{Z}\sum_{r=0}^{Z-1} C_r e^{-2\pi irk/Z}, \quad \Delta\omega_k = \sum_{r=0}^{Z-1} \Delta\mu_r e^{-2\pi irk/Z}.$$

The quantity C_r, according to the definition of aerodynamic influence coefficients, is the generalized aerodynamic force acting on the initial ($k = 0$) blade of the ring at synchronous vibrations of the blades according to a given form φ with identical amplitude and phase shift $2\pi r/Z$ between vibrations of neighboring blades. In the case of a uniform ring ($\Delta\omega_k = 0$), from (12′) we obtain an equation for the eigenvalue $\nu = -C_k$, ($k = 0, 1, \ldots, Z-1$) and a necessary condition for self-excited vibrations $\mathrm{Im}C_k \geqslant 0$, $k = 0, 1, \ldots, Z-1$.

If the blades have a disturbance in the natural frequencies, then the problem reduces to determining the roots of the characteristic equation in system (12) with matrix D or of system (12′) with matrix D'. The matrices D and D' obviously differ by the form of disturbance of the natural frequencies and by the type of nonstationary aerodynamic characteristics of the ring used in the calculation. In particular, a rapid decrease in the value of $|A_r|$ with increase in the number r [1] allows an approximation of matrix D by a band matrix. It is necessary, however, to keep in mind that such an approximation is appropriate only in the case where the critical phase shift μ^* does not occur close to the values $\mu = 0$ and $\mu = 2\pi$.

For real blade rings, the value of Z is often large, and matrix D (or D') is a full matrix of a high order. In addition, the matrix determined from design geometrical parameters and calculation characteristics of a specific blade ring inevitably contains uncontrollable small deviations from that matrix which would be adequate for the blade ring. These deviations are connected with errors in the aerodynamic calculation as well as with technological deviations in a specific blade ring. To these errors must be added approximation errors in the computer, which appeared during operations with a matrix that were necessary for the calculation of characteristic numbers. Thus, any realization of matrix D (or D') connected with a specific blade ring should be considered only as a representative of a set of similar matrices. The practical problem is that all noted deviations together may lead to a significant difference in determination of the flow regime past the ring by the exact criteria $\lambda = \mathrm{Re}\,\lambda$.

The circumstance just described led to an attempt to use, for the stability analysis of vibrations of a blade ring as an aeroelastic system, the stability quality criterion $\chi(D)$ of matrix D (or D') [3, 4], which is determined by the equation

$$\chi(D) = 2\|D\|\max\left[\int_0^\infty \|\mathbf{X}\|^2\, dt/\|\mathbf{X}(0)\|^2\right] \qquad (\chi \in (1, \infty))$$

and is a characteristic of the asymptotic stability of the solution of a system of the type $d\mathbf{X}/dt = D\mathbf{X}$ at some initial disturbance $\mathbf{X}(0)$. The value of $\chi(D)$ slightly depends on small variations of coefficients of the matrix D and can serve as a characteristic of the stability of a blade ring described by the set of matrices represented by matrix D (or D').

Since χ gives an upper bound for the solution at the initial disturbance $\mathbf{X}(0)$, then the operating regimes of a blade ring when $\chi(D) > \chi_b$, where the value of χ_b must be determined empirically, can be classified as practically unstable. In particular, if the allowable level of dynamic stresses in the blades of axial compressors of gas-turbine engines (GTE) is taken to be $(5\text{–}7) \cdot 10^7$ Pa [with the usual level of background tension $(2\text{–}3) \cdot 10^7$ Pa], then $\chi_b = 30\text{–}50$.

The relative technological deviation of the natural frequencies for the first three forms of blade vibrations in aircraft GTE, which can reach up to a few percent, as is known [1, 5], is one of the most important but difficult to evaluate factors influencing the shift of the boundary of blade ring vibrations in the field of its characteristics. The stability criterion $\chi(D)$ can serve as a convenient tool for estimation of the influence of said deviation on the aeroelastic properties of the blade ring.

As an example, Fig. 4 gives the results of calculating the value of $\chi(D')$ as a function of the relative

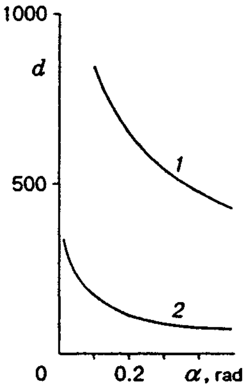

Fig. 3

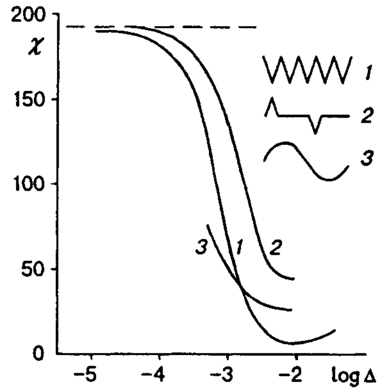

Fig. 4

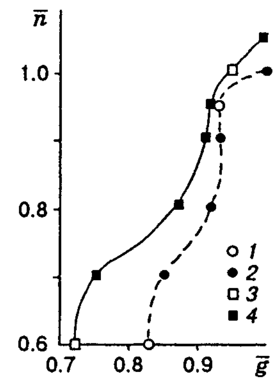

Fig. 5

deviation of frequencies for various types of blade ring designs having slightly different natural frequencies of vibration. The calculation was carried out for a ring of 20 nontwisted blades with profiles of constant height corresponding to the middle cross section of a rotor blade in a compressor stage of a GTE. The flow regime past the ring is chosen to be near the boundary of self-excited vibrations of a corresponding uniform ring with Sh = 0.011 (Sh* = 0.01). The corresponding value of $\chi(D')$ in Fig. 4 is designated by a dashed line.

For calculation of matrix D', a distribution of values of $\Delta\mu_r (r = 0, 1, \ldots, 19)$ was specified, corresponding to one of the forms of disturbance shown in Fig. 4, where the value of $\log \Delta$ (Δ is equal to the mean-square value of $\Delta\omega_k$, $k = 0, 1, \ldots, 19$) is plotted on the x axis.

One can see from the results obtained that, beginning with some value of Δ, a small ($\Delta \sim 10^{-2}$) disturbance of the natural frequencies leads to a sharp decrease in the value of χ, and, in the given regime, provides for practical stability of the blade ring under investigation. Moreover, as noted earlier [6], the ring construction denoted by number 1 ("saw-tooth") in Fig. 1 is the most effective.

The approach to the analysis of the aeroelastic stability of a blade ring described above is taken as a basis for the method of calculating flutter of a blade ring in the space of its parameters. Calculation of forms and frequencies of characteristic vibrations of a design blade in vacuum is carried out by the theory of thin-walled rods [7]. Nonstationary aerodynamic characteristics are calculated by the two-dimensional theory of flow past a cascade of vibrating profiles for subsonic flow regimes [1, 8] or for a cascade of straight segments for supersonic regimes [9]. Consequent calculation of the generalized aerodynamic forces is realized by a numerical integration along the middle surface of the blade. Effects connected with flow in the radial gap of the ring are not taken into account.

As an illustration of the application of the suggested method, Fig. 5 shows the results of calculating the boundary of bending vibrations of the high-pressure fan rotor blading (calculated rate of increase of total pressure ~1.7) of an aircraft GTE using design parameters. On the x axis is plotted the value of mass flow rate $\bar{g}$ normalized to its maximum value at a given rate of rotation $\bar{n}$. Points 1 and 2 refer to the calculation (points 1 are the values of $\bar{g}$ obtained at the point of breakaway, and points 2 occur at the appearance of self-excited blade vibrations of the first characteristic form). Thus, the area to the right of the curve (under $\bar{g} = 1$) shows the range of flow rates reached at the rotor stage with change in rotation rate.

As one can see from the calculations results, for this stage, the self-excited vibrations of the rotor blades of the first form can be expected on the left branches of characteristics for the average rate of rotation. For nominal rotation rates ($\bar{n} = 1$), all points of the characteristic appear in a supersonic flutter zone.

Based on the results obtained, a corresponding modeling stage was constructed in such a way that the natural frequencies of the rotor blades were alternated within a technological deviation, which in this case was $\Delta \sim 0.05$. The results of testing the modeling stage are denoted in Fig. 5 by points 3 and 4 (points 3 refer to the achieved point of breakaway, and 4 refer to regimes of self-excited bending vibrations).

As would be expected, the range of allowable flow rate variation for a real stage increased in comparison with the range predicted by the calculations, although, in the tests, flutter at the medium rotation rates took place before the breakaway. Supersonic flutter appeared only at rotation rates exceeding the nominal ones. Calculation error probably can be explained by the inaccuracy of an aerodynamic model that describes transonic and supersonic flow past the rotor.

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